Riemann-Hilbert problem associated with Angelesco systems

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Abstract

Angelesco systems of measures with Jacobi type weights are considered. For such systems, strong asymptotics for the related multiple orthogonal polynomials are found as well as the Szegő-type functions. In the procedure, an approach from Riemann-Hilbert problem plays a fundamental role.

Key words: approximation by rational function, rate of convergence, simultaneous approximation,
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1 The statement of the Riemann-Hilbert problem

In this work the problem considered is a particular case of the general situation analyzed in [2]. However, due to the simplicity of the case considered, we are able to compute the Szegő-type functions in great detail (cf. (11)).

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Let $\Delta_1 = [-\lambda, -1]$ and $\Delta_2 = [1, \lambda]$ be two intervals on the real line $\mathbb{R}$. For each $j = 1, 2$, take a holomorphic function h_j, on a neighborhood $\mathcal{V}_{h_j}$ of Δ_j, i.e. $h_j \in H(\mathcal{V}_{h_j})$. We also require that such function h_j does not vanishes on $\mathcal{V}_{h_j}$, acquiring only positive values on Δ_j. Observe that $1/h_j \in H(\mathcal{V}_{h_j})$, $j = 1, 2$. Let us define the system of measures (σ_1, σ_2) where σ_1 and σ_2 have the differential form

$$d\sigma_j(x) = \frac{h_j(x)dx}{\sqrt{(\lambda - |x|)(|x| - 1)}}\,,\ x \in \Delta_j,\ j = 1, 2.$$

This system (σ_1, σ_2) belongs to the class of Angelesco systems introduced by Angelesco in [1]. Fix a multi-index $\mathbf{n} = (n_1, n_2)$, we say that a polynomial $Q_{\mathbf{n}} \not\equiv 0$ is a type II multiple-orthogonal polynomial corresponding to a system (σ_1, σ_2), if $\deg Q_{\mathbf{n}} \leq |\mathbf{n}| = n_1 + n_2$ and $Q_{\mathbf{n}}$ satisfies the following orthogonality conditions

$$\int_{\Delta_j} x^\nu Q_{\mathbf{n}}(x) d\sigma_j(x) = 0\,,\ \nu = 0\,, \ldots, n_j - 1,\ j = 1, 2\,. \tag{1}$$

It is well known (see [1]) that for any multi-index $\mathbf{n} = (n_1, n_2)$, the polynomial $Q_{\mathbf{n}}$ has for each $j = 1, 2$, exactly n_j simple zeros lying in the interior set of Δ_j, which we represent by $\mathring{\Delta}_j$. We will denote the function of the second kind

$$R^j_{\mathbf{n}}(z) = \frac{1}{2\pi i} \int_{\Delta_j} Q_{\mathbf{n}}(x) \frac{d\sigma_j(x)}{x - z}\,. \tag{2}$$

Let us take a subset of multi-indices $\Lambda = \{\mathbf{n} = (n, n) : n \in \mathbb{Z}\}$. In the present article we obtain results about the strong asymptotics of the sequence of multi-orthogonal polynomials $\{Q_{\mathbf{n}} : \mathbf{n} \in \Lambda\}$. An effective method for such study with this kind of so "very nice" measures, is analyzing the Riemann-Hilbert problem for multi-orthogonal polynomials, which was introduced in [12]. Let us consider a 3×3 matrix, Y, whose entries are complex functions $Y_{s,k} : \mathbb{C}\setminus(\Delta_1 \cup \Delta_2) \to \mathbb{C}$, $s, k = 1, 2, 3$. Given a point $x \in \mathring{\Delta}_1 \cup \mathring{\Delta}_2$, the following matricial limits, where $z \in \mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$ tending to x, represent the formal pointwise non tangential limits of all entries of Y at the same time:

$$\lim_{z\to x} Y(z) = Y_+(x)\,, \Im m\,(z) > 0 \quad \text{and} \quad \lim_{z\to x} Y(z) = Y_-(x)\,, \Im m\,(z) < 0\,.$$

Let $\delta_{s,k} : \mathbb{N}^2 \to \{0, 1\}$ denote the Kronecker delta function, i.e. $\delta_{s,k} = 0$ when $s \neq k$, and $\delta_{s,s} = 1$, $s, k \in \mathbb{N}$. Let us look for a matrix function Y, which satisfies the following conditions:

(1) The entries of Y, $Y_{s,k}$, belongs to $H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$, which we write as $Y \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$;

(2) For each Δ_j, $j=1,2$, the so called jump condition takes place

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \frac{\delta_{1,j} h_1(x)}{\sqrt{(\lambda-|x|)(1-|x|)}} & \frac{\delta_{2,j} h_2(x)}{\sqrt{(\lambda-|x|)(1-|x|)}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ x \in \mathring{\Delta}_j\,;$$

(3) Given a multi-index $\mathbf{n}=(n,n)\in\Lambda$, we require the following asymptotic condition at infinity,

$$Y(z) \begin{pmatrix} z^{-2n} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^n \end{pmatrix} = \mathbb{I} + \mathcal{O}\left(1/z\right) \quad \text{as} \quad z\to\infty,$$

where $\mathbb{I}$ is the identity matrix of size 3×3;

(4) For each $i,j=1,2$, we set the following behavior around the endpoints $c_{1,1}=-\lambda$, $c_{2,1}=-1$, $c_{1,2}=1$ and $c_{2,2}=\lambda$,

$$Y(z) = \mathcal{O}\begin{pmatrix} 1 & \delta_{2,j} + \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} & \delta_{1,j} + \frac{\delta_{2,j}}{\sqrt{|z-c_{i,j}|}} \\ 1 & \delta_{2,j} + \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} & \delta_{1,j} + \frac{\delta_{2,j}}{\sqrt{|z-c_{i,j}|}} \\ 1 & \delta_{2,j} + \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} & \delta_{1,j} + \frac{\delta_{2,j}}{\sqrt{|z-c_{i,j}|}} \end{pmatrix}, \quad \text{as} \quad z\to c_{i,j}\,.$$

This problem, which consists in finding the matrix function Y, was called in [12] a Riemann-Hilbert problem for type II multiple orthogonal polynomials, and for the system of measures (σ_1,σ_2), RHP in short. The solution Y is unique and has the form

$$Y(z) = \begin{pmatrix} Q_{\mathbf{n}}(z) & R^1_{\mathbf{n}}(z) & R^2_{\mathbf{n}}(z) \\ d_1 Q_{\mathbf{n}^1_-}(z) & d_1 R^1_{\mathbf{n}^1_-}(z) & d_1 R^2_{\mathbf{n}^1_-}(z) \\ d_2 Q_{\mathbf{n}^2_-}(z) & d_2 R^1_{\mathbf{n}^2_-}(z) & d_2 R^2_{\mathbf{n}^2_-}(z) \end{pmatrix}, \tag{3}$$

with

$$d_j^{\,-1} = -\frac{1}{2\pi i}\int_{\Delta_j} x^{n-1} Q_{\mathbf{n}^j_-}(x) d\sigma_j(x)\,,$$

where $\mathbf{n}^1_- = (n-1,n)$ and $\mathbf{n}^2_-=(n,n-1)$.

The key of our procedure is inspired in the works [2,3,9,10] and it is based in finding the relationship between Y and a matrix function R which is the solution of the following RHP:

(1) $R:\mathbb{C}\to\mathbb{C}^{3\times 3}$ belongs to $H(\mathbb{C}\setminus\gamma)$;

(2) $R_+(\xi) = R_-(\xi)V_n(\xi)$, $\xi \in \gamma$;
(3) $R(z) \to \mathbb{I}$ as $z \to \infty$;

where $V_n \in H(\mathcal{A})$, with $\mathcal{A} \subset \mathbb{C}$ a certain domain, $V_n = \mathbb{I} + \epsilon_n$, such that $\epsilon_n \to 0$ uniformly on compact subsets of $\mathcal{A}$ as $n \to \infty$, and γ is a contour or system of contours, that is contained in $\mathcal{A}$. In this case we can assure that

$$R = \mathbb{I} + \mathcal{O}\left(\epsilon_n\right) \,.$$

The RHP for Y is not normalized in the sense that the conditions (3) at infinity for Y and R are different. In order to normalize the RHP, we are going to modify Y in such a way that we set another RHP with the same contours (possibly different jump conditions), for which the solution tends to the identity matrix as $z \to \infty$. For normalizing we need to take into account the behavior of $Y(z)$ for large z. This behavior depends on the distribution of the zeros of the multiple-orthogonal polynomials. The zero distribution of the orthogonal polynomials is usually given by an extremal problem in logarithmic potential theory. In section 2 we introduce some concepts and results which we will need about this theory and we will normalize the Riemann-Hilbert problem at infinity. In section 3 such a Riemann-Hilbert problem with oscillatory and exponentially decreasing jumps can be analyzed by using the steepest descent method introduced by Deift and Zhou (see [5,6]). The first work such that the orthogonal polynomials appear as solution of a Riemann-Hilbert problem is [7], and in [4] these ideas were for the first time applied to get strong asymptotics for orthogonal polynomials.

2 The equilibrium problem and the normalization at infinity

Let us fix $j \in \{1, 2\}$. $\mathcal{M}_{1/2}(\Delta_j)$ denotes the set of all finite Borel measures whose supports, i.e. supp $(\cdot)$, are contained in Δ_j with total variation $1/2$. Take $\mu_j \in \mathcal{M}_{1/2}(\Delta_j)$ and define its logarithmic potential as follows

$$V^{\mu_j}(z) = \int \log \frac{1}{|z-x|} d\mu_j(x) \,, \quad z \in \mathbb{C} \,.$$

For each pair of measures (μ_1, μ_2), where $\mu_j \in \mathcal{M}_{1/2}(\Delta_j)$, $j = 1, 2$, we define the quantities

$$m_j(\mu_1, \mu_2) = \min_{x \in \Delta_j} \left(2V^{\mu_j}(x) + V^{\mu_k}(x)\right) \,, \; j, k = 1, 2, \; j \neq k \,.$$

The following Proposition is deduced immediately from the results of [8].

Proposition 1 *There exists a unique pair* $(\bar{\mu}_1, \bar{\mu}_2) \in \mathcal{M}_{1/2}(\Delta_1) \times \mathcal{M}_{1/2}(\Delta_2)$, *which satisfies for* $j,k=1,2$

$$2V^{\bar{\mu}_j}(x)+V^{\bar{\mu}_k}(x)=m_j(\bar{\mu}_1,\bar{\mu}_2)=m_j,\ \ x\in \operatorname{supp}(\bar{\mu}_j)=\Delta_j\,,\ \ j\neq k\,.$$

For each $j=1,2$ *the measure* $\bar{\mu}_j$ *is absolutely continuous and has the following differential form*

$$d\bar{\mu}_1(x)=\frac{\rho_1(x)dx}{\sqrt{(\lambda-|x|)(|x|-1)}},\quad d\bar{\mu}_2(x)=\frac{\rho_2(x)dx}{\sqrt{(\lambda-|x|)(|x|-1)}}\,,$$

where ρ_j *is a function which has an analytic continuation to a neighborhood* $\mathcal{V}_{\rho_j}$ *of the interval* Δ_j.

In what follows we consider $\mathcal{V}_j=\mathcal{V}_{h_j}\cap\mathcal{V}_{\rho_j}$. The pair $(\bar{\mu}_1,\bar{\mu}_2)$ is called extremal or equilibrium pair of measures with respect to (Δ_1,Δ_2). Let us denote for each $j=1,2$ the analytic potentials

$$g_j(z)=\int_{\Delta_j}\log(z-x)d\bar{\mu}_j(x)=-V^{\bar{\mu}_j}(z)+i\int_{\Delta_j}\arg(z-x)d\mu_j(x)\,,$$

where arg denotes the principal argument function.
Substituting the logarithmic potential in Proposition 1 we obtain for each $j,k=1,2$ with $j\neq k$ that

$$-(g_{j+}(x)+g_{j-}(x))-g_{k-}(x)=m_j\,,\quad x\in\Delta_j\,.$$

Observe that if $c_{1,1}=-\lambda$, $c_{2,1}=-1$, $c_{1,2}=1$ and $c_{2,2}=\lambda$, then

$$g_{j+}(x)-g_{j-}(x)=\begin{cases}0 & \text{if}\quad c_{2,j}\leq x\\ i\pi & \text{if}\quad c_{1,j}\geq x\\ 2i\pi\int_x^{c_{2,j}}d\bar{\mu}_j(t) & \text{if}\quad x\in\Delta_j\end{cases}.$$

Let us introduce the matrices

$$G(z)=\begin{pmatrix}e^{-n(g_1(z)+g_2(z))} & 0 & 0\\ 0 & e^{ng_1(z)} & 0\\ 0 & 0 & e^{ng_2(z)}\end{pmatrix}\quad\text{and}\quad L=\begin{pmatrix}1 & 0 & 0\\ 0 & e^{-nm_1} & 0\\ 0 & 0 & e^{-nm_2}\end{pmatrix}.\tag{4}$$

We define the matrix function $T=LYGL^{-1}$, where L,G are as in (4) and Y is given by (3). Hence T is the unique solution of the RHP:

(1) $T\in H(\mathbb{C}\setminus(\Delta_1\cup\Delta_2))$;
(2) $T_+(x)=T_-(x)M(x)$, $x\in\mathring{\Delta}_1\cup\mathring{\Delta}_2$;
(3) $T(z)=\mathbb{I}+\mathcal{O}(1/z)$ as $z\to\infty$;
(4) T and Y have the same behavior on the endpoints of the intervals Δ_j, for $j=1,2$;

where the jump matrix M has the form

$$M(x)=\begin{pmatrix} e^{-2ni\pi\int_x^{c_{2,j}}d\bar{\mu}_j(t)} & \frac{\delta_{j,1}h_1(x)}{\sqrt{(\lambda-|x|)(|x|-1)}} & \frac{\delta_{j,2}h_2(x)dx}{\sqrt{(\lambda-|x|)(|x|-1)}} \\ 0 & e^{2n\delta_{j,1}i\pi\int_x^{c_{2,1}}d\bar{\mu}_1(t)} & 0 \\ 0 & 0 & e^{2n\delta_{j,2}i\pi\int_x^{c_{2,2}}d\bar{\mu}_2(t)} \end{pmatrix}, \qquad (5)$$

with $x\in\mathring{\Delta}_j$.

3 The opening of the lens

Let us consider

$$\phi_1(z)=-\pi\int_z^{-1}\frac{\rho_1(\zeta)d\zeta}{\sqrt{(\zeta+\lambda)(\zeta+1)}},\ z\in\mathcal{V}_1$$

and

$$\phi_2(z)=-\pi\int_z^{\lambda}\frac{\rho_2(\zeta)d\zeta}{\sqrt{(\zeta-\lambda)(\zeta-1)}},\ z\in\mathcal{V}_2\,.$$

We have considered $\sqrt{(\zeta+\lambda)(\zeta+1)}$ and $\sqrt{(\zeta-\lambda)(\zeta-1)}$ as analytic functions on $\mathbb{C}\setminus\Delta_1$ and $\mathbb{C}\setminus\Delta_2$, respectively, where we have taken the branches which are positive for real $\zeta>-1$ and $\zeta>\lambda$, respectively. Observe that for each $j=1,2$, the function $\phi_j\in H(\mathcal{V}_j\setminus\Delta_j)$, the real part of the functions $\phi_{j\pm}$ vanish on Δ_j, $\Re e\,(\phi_{j\pm})(x)=0$, $x\in\Delta_j$, and their derivatives

$$\phi'_{j\pm}(x)=\mp i\pi\frac{\rho_j(x)}{\sqrt{(\lambda-|x|)(|x|-1)}}\,.$$

By the Cauchy-Riemann conditions we have that

$$\pm\frac{\partial\Re e\ \phi_{\pm}}{\partial y}(x)>0\,,\ x\in\Delta_j\,.$$

Since $\Re e\,\phi_j$ is a harmonic function on $\mathcal{V}_j\setminus\Delta_j$ we can assure that $\Re e\,\phi_j(z)>0$, $z\in\mathcal{V}_j\setminus\Delta_j$.

Factorize the jump matrix function M in (5) as follows

$$M(x)=\begin{pmatrix} 1 & 0\ 0 \\ \frac{\delta_{j,1}e^{-2n\phi_{1-}(x)}\sqrt{(\lambda-|x|)(|x|-1)}}{h_1(x)} & 1\ 0 \\ \frac{\delta_{j,2}e^{-2n\phi_{2-}(x)}\sqrt{(\lambda-|x|)(|x|-1)}}{h_2(x)} & 0\ 1 \end{pmatrix}$$

$$
\times \begin{pmatrix} 0 & \frac{\delta_{j,1}h_1(x)}{\sqrt{(\lambda-|x|)(|x|-1)}} & \frac{\delta_{j,2}h_2(x)}{\sqrt{(\lambda-|x|)(|x|-1)}} \\ -\frac{\delta_{1,j}\sqrt{(\lambda-|x|)(|x|-1)}}{h_1(x)} & \delta_{j,2} & 0 \\ -\frac{\delta_{2,j}\sqrt{(\lambda-|x|)(|x|-1)}}{h_2(x)} & 0 & \delta_{j,1} \end{pmatrix}
$$

$$
\times \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_{j,1}e^{-2n\phi_{1+}(x)}\sqrt{(\lambda-|x|)(|x|-1)}}{h_1(x)} & 1 & 0 \\ \frac{\delta_{j,2}e^{-2n\phi_{2+}(x)}\sqrt{(\lambda-|x|)(|x|-1)}}{h_2(x)} & 0 & 1 \end{pmatrix}.
$$

Now we are going to follow a procedure analogous to the one in [3]. For each $j = 1,2$ let us fix a closed curve γ_j contained in $\mathcal{V}_j$, with the clockwise orientation. Set Γ_j the bounded connected component of $\mathbb{C}\setminus\gamma_j$. Let us introduce the matrix function S, defined by

$$
S(z) = T(z)\begin{pmatrix} 1 & 0 & 0 \\ \frac{i\delta_{1,j}e^{-2n\phi_1(z)}\sqrt{(z+\lambda)(z+1)}}{h_1(z)} & 1 & 0 \\ \frac{i\delta_{2,j}e^{-2n\phi_2(z)}\sqrt{(z-\lambda)(z-1)}}{h_2(z)} & 0 & 1 \end{pmatrix}, \; z \in \Gamma_j \, ,
$$

and $S(z) = T(z)\,,\ z \in \mathbb{C}\setminus\overline{\Gamma}_j$.

The matrix function S satisfies the RHP:

(1) $S \in H(\mathbb{C}\setminus\cup_{j=1,2}(\Delta_j\cup\gamma_j))$;
(2) The jump conditions $j = 1,2$ are,

$$
S_+(x) = S_-(x)\begin{pmatrix} 0 & \frac{\delta_{1,j}h_1(x)}{\sqrt{(\lambda-|x|)(|x|-1)}} & \frac{\delta_{2,j}h_2(x)}{\sqrt{(\lambda-|x|)(|x|-1)}} \\ -\frac{\delta_{1,j}\sqrt{(\lambda-|x|)(|x|-1)}}{h_1(x)} & \delta_{2,j} & 0 \\ -\frac{\delta_{2,j}\sqrt{(\lambda-|x|)(|x|-1)}}{h_2(x)} & 0 & \delta_{1,j} \end{pmatrix},
$$

when $x \in \mathring{\Delta}_j$, and if $z \in \gamma_j$,

$$
S_+(z) = S_-(z)\begin{pmatrix} 1 & 0 & 0 \\ \frac{i\delta_{1,j}e^{-2n\phi_1(z)}\sqrt{(z+\lambda)(z+1)}}{h_1(z)} & 1 & 0 \\ \frac{i\delta_{2,j}e^{-2n\phi_2(z)}\sqrt{(z-\lambda)(z-1)}}{h_2(z)} & 0 & 1 \end{pmatrix};
$$

(3) $S(z) = \mathbb{I} + \mathcal{O}(1/z)$ as $z \to \infty$;
(4) The conditions for the endpoints are the same as for T.

Now, we consider the limiting problem, because for the matrix S the jump matrix function on each γ_j for $j = 1,2$ tends to the identity matrix when

$n \to \infty$. We look for the matrix function N which satisfies the following RHP:

(1) $N \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$;
(2) The jump conditions in $\overset{\circ}{\Delta}_j$ for $j = 1, 2$ are,

$$N_+(x) = N_-(x) \begin{pmatrix} 0 & \frac{\delta_{1,j} h_1(x)}{\sqrt{(\lambda-|x|)(|x|-1)}} & \frac{\delta_{2,j} h_2(x)}{\sqrt{(\lambda-|x|)(|x|-1)}} \\ -\frac{\delta_{1,j}\sqrt{(\lambda-|x|)(|x|-1)}}{h_1(x)} & \delta_{2,j} & 0 \\ -\frac{\delta_{2,j}\sqrt{(\lambda-|x|)(|x|-1)}}{h_2(x)} & 0 & \delta_{1,j} \end{pmatrix}; \quad (6)$$

(3) $N(z) = \mathbb{I} + \mathcal{O}(1/z)$ as $z \to \infty$;
(4) N satisfies the same conditions for the endpoints as S.

Let us consider the matrix function $K = [K_{k,l}]$, $k, l = 1, 2, 3$ that is the solution of the RHP:

(1) $K \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$;
(2) The jump conditions in $\overset{\circ}{\Delta}_j$ for $j = 1, 2$ are, because of (6),

$$K_+(x) = K_-(x) \begin{pmatrix} 0 & \frac{\delta_{1,j}}{\sqrt{(\lambda-|x|)(|x|-1)}} & \frac{\delta_{2,j}}{\sqrt{(\lambda-|x|)(|x|-1)}} \\ -\delta_{1,j}\sqrt{(\lambda-|x|)(|x|-1)} & \delta_{2,j} & 0 \\ -\delta_{2,j}\sqrt{(\lambda-|x|)(|x|-1)} & 0 & \delta_{1,j} \end{pmatrix} \quad (7)$$

(3) $K(z) = \mathbb{I} + \mathcal{O}(1/z)$ as $z \to \infty$;
(4) The conditions for the endpoints are the same as for N.

Notice that when $h_j = 1$, $j = 1, 2$, K and N have the same RHP. Analogously to the ideas in [3], let us again consider $\sqrt{(z+\lambda)(z+1)}$ and $\sqrt{(z-\lambda)(z-1)}$ as analytic functions on $\mathbb{C} \setminus \Delta_1$ and $\mathbb{C} \setminus \Delta_2$, respectively, where we have taken the branches which are positive for real $z > -1$ and $z > \lambda$, respectively,

$$\left(\frac{1}{i}\sqrt{(z+\lambda)(z+1)}\right)_\pm (x) = \pm\sqrt{(\lambda+x)(-x-1)}\,, \quad x \in \overset{\circ}{\Delta}_1$$

and

$$\left(\frac{1}{i}\sqrt{(z-\lambda)(z-1)}\right)_\pm (x) = \pm\sqrt{(\lambda-x)(x-1)}\,, \quad x \in \overset{\circ}{\Delta}_2\,.$$

For each $k = 1, 2, 3$, we rewrite (7) as

$$\begin{cases} \left(\frac{1}{i}\sqrt{(z+\lambda)(z+1)}\,K_{k,2}\right)_\pm (x) = (K_{k,1})_\mp(x) \\ (K_{k,3})_+(x) = (K_{k,3})_-(x) \end{cases}, \quad x \in \overset{\circ}{\Delta}_1$$

$$\begin{cases} \left(\frac{1}{i}\sqrt{(z-\lambda)(z-1)}\,K_{k,3}\right)_{\pm}(x) = (K_{k,1})_{\mp}(x) \\ (K_{k,2})_{+}(x) = (K_{k,2})_{-}(x) \end{cases} , \; x \in \overset{\circ}{\Delta}_2$$

and we denote

$$\psi_0^k(z) = K_{k,1}(z)\,,\; \psi_1^k(z) = \frac{1}{i}\sqrt{(z+\lambda)(z+1)}K_{k,2}(z)$$

$$\text{and } \psi_2^k(z) = \frac{1}{i}\sqrt{(z-\lambda)(z-1)}K_{k,3}(z)\,.$$

Then from the relations (7), we may interpret each row $k = 1, 2, 3$ of such matrix K as a function defined on a Riemann surface. Let $\mathcal{R}$ define the Riemann surface which has two cuts. One of them connects the two branch points $-\lambda$ and -1 with the cut in the interval Δ_1. The other cut is made in the interval Δ_2, to connect the two other branch points 1 and λ. The sheet $\mathcal{R}_0$ is glued to another sheet $\mathcal{R}_1$ along the cut Δ_1, and $\mathcal{R}_0$ is also glued to $\mathcal{R}_2$ along the interval Δ_2. Let us denote by ψ^k, $k = 1, 2, 3$, three multi-valued functions $\psi^k = \left(\psi_0^k, \psi_1^k, \psi_2^k\right)$, such that for each $k = 1, 2, 3$ its components ψ_l^k, $l = 0, 1, 2$, $k = 1, 2, 3$, map the corresponding sheet $\mathcal{R}_l$ onto $\mathbb{C}$, and satisfy:

i) $\psi_0^k \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$, $\psi_j^k \in H(\mathbb{C} \setminus \Delta_j)$, $j = 1, 2$;
ii) $\psi_{0\pm}^k = \psi_{j\mp}^k$, $j = 1, 2$;
iii) $\psi_0^k = \delta_{k,1} + \mathcal{O}(1/z)$, and $\psi_l^k(z) = -iz\delta_{k,l+1} + \mathcal{O}(1)$, $l = 1, 2$, as $z \to \infty$;
iv) $\psi_l^k(z) = \mathcal{O}(1)$, at the endpoints.

Observe that $\psi^1 : \mathcal{R} \to \mathbb{C}$ is a bounded holomorphic function on $\mathcal{R}$, where $\lim_{z\to\infty} \psi_0^1(z) := \psi_0^1(\infty) = 1$. This implies that ψ^1 is the constant function identically equal to 1, i.e. $\psi^1 \equiv 1$. For the cases when $k = 2, 3$, G. López Lagomasino et al., [11], proved that up to complex constants c_1, c_2

$$\psi^2(z) = \frac{c_1}{\varphi(z)} \quad \text{and} \quad \psi^3(z) = c_2\frac{\varphi^1(z)}{\varphi(z)}\,,$$

where

$$\varphi(z) = \left(\frac{1+a^2}{(1-a^2)^2}\right)^{1/3}\left(1 + G^{-1}(z)\right)\,,\;\; \varphi^1(z) = \frac{1+G^{-1}(z)}{1-G^{-1}(z)}, \tag{8}$$

$$G(w) = \frac{H(w)}{H(a)}\,,\;\; H(w) = w - \frac{(1-a^2)^2 w}{(1+a^2)(1-w^2)}\,,$$

and a is the unique solution on the interval $]0, 1[$ of the biquartic equation

$$a^8 + (16\lambda^2 - 8)a^6 + 18a^4 - 27 = 0\,. \tag{9}$$

In this case, $H^{-1}(z)$ is the solution of the cubic equation

$$w^3 - zw^2 + \frac{a^4 - 3a^2}{1+a^2}w + z = 0\,. \tag{10}$$

Notice that given a value $\lambda > 1$, the equation (9) as well as (10) can be solved by elementary methods.

Let us find the diagonal 3×3 matrix function $D = \operatorname{diag}(D_0, D_1, D_2)$, such that $N(z) = D^{-1}(\infty)K(z)D(z)$. The conditions (7) imply that the entries of D must satisfy the following conditions

$$h_j(x)D_{0\pm}(x) = D_{j\mp}(x)\,,\ D_{k+}(x) = D_{k-}(x) \text{ when } x \in \overset{\circ}{\Delta}_j\,,\ j,k = 1,2\,,\ k \neq j\,,$$

i.e. D_l, $l = 0,1,2$ are the *Szegő-type functions.*
Analogously to the function ψ_j^i, we obtain the following problem for the entries of D:

i) $D_0 \in H(\overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2))$, $D_j \in H(\overline{\mathbb{C}} \setminus \Delta_j)$, $j = 1,2$;
ii) $h_j(x)D_{0\pm}(x) = D_{j\mp}(x)$, $j = 1,2$;
iii) $D_l(z) = \mathcal{O}(1)$, $l = 0,1,2$, at the endpoints.

In order to find this matrix function D, we consider the function φ given by (8), such that its components φ_l, $l = 0,1,2$, map the corresponding sheet $\mathcal{R}_l$ on $\mathbb{C}$, and satisfy:

i) $\varphi_0 \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$, $\varphi_j \in H(\mathbb{C} \setminus \Delta_j)$, $j = 1,2$;
ii) $\varphi_{0\pm} = \varphi_{j\mp}$, $j = 1,2$;
iii) $\varphi_0(z) = \mathcal{O}(z)$, $\varphi_1(z) = \mathcal{O}(1/z)$, and $\varphi_2(z) = \mathcal{O}(1)$, as $z \to \infty$;
iv) $\varphi_0\varphi_1\varphi_2(\infty) = 1$;
v) $\varphi_l(z) = \mathcal{O}(1)$, at the endpoints.

We denote by $\Sigma_j = \varphi_{0-}(\Delta_j) \cup \varphi_{0+}(\Delta_j)$, for $j = 1,2$ the closed curves in the complex plane, with the clockwise orientation, and we denote by Ω_j the interior set of Σ_j for $j = 0,1,2$ and by Ω_0 the exterior set of $\Sigma_1 \cup \Sigma_2$. Taking into account the behavior of the functions φ_l at infinity, $\Omega_l = \varphi_l(\mathcal{R}), l = 0,1,2$. Using (10) we get that $\varphi(z)$ is the solution of the cubic algebraic equation

$$w^3 - \left(\frac{1+a^2}{(1-a^2)^2}\right)^{1/3}(3+z)w^2 + \left(\frac{1+a^2}{(1-a^2)^2}\right)^{2/3}\left(2z + \frac{3+a^4}{1+a^2}\right)w - 1 = 0\,,$$

that is equivalent to

$$z = \frac{w^3 - 3\left(\frac{1+a^2}{(1-a^2)^2}\right)^{1/3} w^2 + \left(\frac{1+a^2}{(1-a^2)^2}\right)^{2/3} \frac{3+a^4}{1+a^2} w - 1}{\left(\frac{1+a^2}{(1-a^2)^2}\right)^{1/3} w^2 - 2\left(\frac{1+a^2}{(1-a^2)^2}\right)^{2/3} w} =: r(w)\,.$$

Using this rational function r we consider the complex function $\tilde{D}$, defined as

$$\tilde{D}(w) = \begin{cases} D_0(r(w))\,, & w \in \Omega_0 \\ D_1(r(w))\,, & w \in \Omega_1 \\ D_2(r(w))\,, & w \in \Omega_2 \end{cases} .$$

This function $\tilde{D}$ verifies the multiplicative scalar Riemann-Hilbert problem

$$h_j(r(\xi))\tilde{D}(\xi)_- = \tilde{D}(\xi)_+, \ \xi \in \Sigma_j, \ j = 1,2\,.$$

Taking into account that $D_0D_1D_2$ is an entire function, and using the behavior at $z = \infty$, it follows that $D_0D_1D_2 \equiv c$, where c is a complex constant. We can choose a single valued branch of the complex logarithm, and we have the additive scalar Riemann-Hilbert problem

$$\log h_j(r(\xi)) + \log \tilde{D}(\xi)_- = \log \tilde{D}(\xi)_+, \ \xi \in \Sigma_j, \ j = 1,2\,.$$

Using the Sokhotsky-Plemelj formula we obtain that

$$\log \tilde{D}(w) = \frac{1}{2\pi i} \sum_{j=1,2} \int_{\Sigma_j} \frac{\log h_j(r(\xi))}{\xi - w} d\xi\,,$$

and so, the Szegő-type functions, are given explicitly by,

$$D_l(z) = \exp\left\{ \frac{1}{2\pi i} \sum_{j=1,2} \varepsilon_j \int_{\Delta_j} \log h_j(x) \left(\frac{-\varphi_{0+}'(x)}{\varphi_{0+}(x) - \varphi_l(z)} \right.\right. \\ \left.\left. + \frac{\varphi_{0-}'(x)}{\varphi_{0-}(x) - \varphi_l(z)} \right) dx \right\}, \quad (11)$$

for $l = 0,1,2$, where $\varepsilon_j = 1$ if orientation of $-\varphi_{0+}(\Delta_j) \cup \varphi_{0-}(\Delta_j)$ is in the clockwise direction, where we are considering that the intervals Δ_j, $j = 1,2$ are oriented from left to right, and $\varepsilon_j = -1$ if this not happen.

For this functions D_l the behavior at the end points of the intervals Δ_j, for $j = 1,2$ is $\mathcal{O}(1)$ if we take into account the quadratic ramifications at these points suggested by the Riemann surface, $\mathcal{R}$.

Finally the matrix function N has the form

$$N(z) = \begin{pmatrix} \frac{D_0(z)}{D_0(\infty)} & \frac{i\,D_1(z)}{D_0(\infty)\sqrt{(z+\lambda)(z+1)}} & \frac{i\,D_2(z)}{D_0(\infty)\sqrt{(z-\lambda)(z-1)}} \\ \frac{D_0(z)\psi_0^2(z)}{D_1(\infty)} & \frac{i\,D_1(z)\psi_1^2(z)}{D_1(\infty)\sqrt{(z+\lambda)(z+1)}} & \frac{i\,D_2(z)\psi_2^2(z)}{D_1(\infty)\sqrt{(z-\lambda)(z-1)}} \\ \frac{D_0(z)\psi_0^3(z)}{D_2(\infty)} & \frac{i\,D_1(z)\psi_1^3(z)}{D_2(\infty)\sqrt{(z+\lambda)(z+1)}} & \frac{i\,D_2(z)\psi_2^3(z)}{D_2(\infty)\sqrt{(z-\lambda)(z-1)}} \end{pmatrix}.$$

Set $R(z) = S(z)N^{-1}(z)$. Since S and N have the same jump across $\overset{\circ}{\Delta}_j$, $j = 1,2$, we have that $R_+(x) = R_-(x)$ for $x \in \overset{\circ}{\Delta}_j$, $j = 1,2$. From the definition of R, and the endpoint conditions for N, we can also deduce that these endpoints are removable singularities. Hence R is an analytic function across the full intervals Δ_1 and Δ_2, and it has jumps on the curves γ_j, $j = 1, 2$. Then we have the following RHP for R:

(1) $R \in H(\mathbb{C} \setminus (\gamma_1 \cup \gamma_2))$;
(2) The jump conditions are for $j = 1, 2$

$$R_+(z) = R_-(z)\, N(z) \begin{pmatrix} 1 & 0 & 0 \\ \frac{i\,\delta_{1,j} e^{-2n\phi_1(z)} \sqrt{(z+\lambda)(z+1)}}{h_1(z)} & 1 & 0 \\ \frac{i\,\delta_{2,j} e^{-2n\phi_2(z)} \sqrt{(z-\lambda)(z-1)}}{h_2(z)} & 0 & 1 \end{pmatrix} N^{-1}(z) \quad \text{if} \quad z \in \gamma_j\,;$$

(3) $R(z) = \mathbb{I} + \mathcal{O}(1/z)$.

Then in each compact $\mathcal{K} \subset \mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$, using the same argument as in [10], we have that $R = \mathbb{I} + \mathcal{O}(e^{-cn})$, with $c(\mathcal{K}) > 0$ uniformly as $n \to \infty$, so it holds uniformly in compact sets of the indicated region that

$$Y(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{nm_1} & 0 \\ 0 & 0 & e^{nm_2} \end{pmatrix} \left(\mathbb{I} + \mathcal{O}\left(e^{-cn}\right)\right) N(z)$$
$$\times \begin{pmatrix} e^{n(g_1(z)+g_2(z))} & 0 & 0 \\ 0 & e^{-n(m_1+g_1(z))} & 0 \\ 0 & 0 & e^{-n(m_2+g_2(z))} \end{pmatrix},$$

$z \in \mathbb{C} \setminus (\overline{\Gamma}_1 \cup \overline{\Gamma}_2)$, and

$$Y(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{nm_1} & 0 \\ 0 & 0 & e^{nm_2} \end{pmatrix} \left(\mathbb{I} + \mathcal{O}\left(e^{-cn}\right)\right) N(z)$$
$$\times \begin{pmatrix} 1 & 0 & 0 \\ \frac{-i\delta_{1,j} e^{-2n\phi_1(z)} \sqrt{(z+\lambda)(z+1)}}{h_1(z)} & 1 & 0 \\ \frac{-i\delta_{2,j} e^{-2n\phi_2(z)} \sqrt{(z-\lambda)(z-1)}}{h_2(z)} & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_1(z)+g_2(z))} & 0 & 0 \\ 0 & e^{-n(m_1+g_1(z))} & 0 \\ 0 & 0 & e^{-n(m_2+g_2(z))} \end{pmatrix},$$

$z \in \Gamma_j$, where N is given by (3).

Finally, we state the main result of this paper.

Theorem 1 *The type II multiple orthogonal polynomial given by* (1)*, has on any compact* $\mathcal{K} \subset \mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$*, uniformly as* $n \to \infty$*, the following strong asymptotic behavior,*

$$
\begin{aligned}
Q_{\mathbf{n}}(z) &= \frac{D_0(z)}{D_0(\infty)} e^{n(g_1(z)+g_2(z))} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) , \\
d_1 Q_{\mathbf{n}^1_-}(z) &= \frac{D_0(z)}{D_0(\infty)} \psi_0^2(z) e^{n(m_1+g_1(z)+g_2(z))} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) , \\
d_2 Q_{\mathbf{n}^2_-}(z) &= \frac{D_0(z)}{D_0(\infty)} \psi_0^3(z) e^{n(m_2+g_1(z)+g_2(z))} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) ,
\end{aligned}
$$

and also holds on any compact $\mathcal{K} \subset \Delta_j\,,\ j,k = 1,2\,,\ j \neq k$*,*

$$
\begin{aligned}
Q_{\mathbf{n}}(x) &= \left\{ \frac{D_{0+}(x)}{D_0(\infty)} e^{ng_{j+}(x)} + \frac{D_{0-}(x)}{D_0(\infty)} e^{ng_{j-}(x)} \right\} e^{ng_k(x)} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) , \\
d_1 Q_{\mathbf{n}^1_-}(x) &= \left\{ \frac{D_{0+}(x)}{D_1(\infty)} e^{ng_{j+}(x)} \psi_{0+}^2(x) + \frac{D_{0-}(x)}{D_1(\infty)} e^{ng_{j-}(x)} \psi_{0-}^2(x) \right\} \\
&\qquad \times e^{n(g_k(x)+m_1)} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) , \\
d_2 Q_{\mathbf{n}^2_-}(x) &= \left\{ \frac{D_{0+}(x)}{D_2(\infty)} e^{ng_{j+}(x)} \psi_{0+}^3(x) + \frac{D_{0-}(x)}{D_2(\infty)} e^{ng_{j-}(x)} \psi_{0-}^3(x) \right\} \\
&\qquad \times e^{n(g_k(x)+m_2)} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) .
\end{aligned}
$$

We can also state:

Theorem 2 *The second kind function given by* (2)*, has on any compact* $\mathcal{K}$ *of the indicated region, uniformly as* $n \to \infty$*, the following strong asymptotic behavior,*

$$
\begin{aligned}
R^1_{\mathbf{n}}(z) &= \frac{i\, D_1(z) e^{-n(m_1+g_1(z))}}{D_0(\infty)\sqrt{(z+\lambda)(z+1)}} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) ,\ z \in \mathbb{C} \setminus \Delta_1 , \\
R^2_{\mathbf{n}}(z) &= \frac{i\, D_2(z) e^{-n(m_2+g_2(z))}}{D_0(\infty)\sqrt{(z-\lambda)(z-1)}} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) ,\ z \in \mathbb{C} \setminus \Delta_2 , \\
d_1 R^1_{\mathbf{n}^1_-}(z) &= \frac{i\, D_1(z) \psi_1^2(z) e^{-ng_1(z)}}{D_1(\infty)\sqrt{(z+\lambda)(z+1)}} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) ,\ z \in \mathbb{C} \setminus \Delta_1 , \\
d_1 R^2_{\mathbf{n}^1_-}(z) &= \frac{i\, D_2(z) \psi_2^2(z) e^{-n(m_2-m_1+g_2(z))}}{D_1(\infty)\sqrt{(z-\lambda)(z-1)}} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) ,\ z \in \mathbb{C} \setminus \Delta_2 , \\
d_2 R^1_{\mathbf{n}^2_-}(z) &= \frac{i\, D_1(z) \psi_1^3(z) e^{-n(m_1-m_2+g_1(z))}}{D_2(\infty)\sqrt{(z+\lambda)(z+1)}} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) ,\ z \in \mathbb{C} \setminus \Delta_1 , \\
d_2 R^1_{\mathbf{n}^2_-}(z) &= \frac{i\, D_2(z) \psi_2^3(z) e^{-ng_2(z)}}{D_2(\infty)\sqrt{(z-\lambda)(z-1)}} \left(1 + \mathcal{O}\left(e^{-cn}\right)\right) ,\ z \in \mathbb{C} \setminus \Delta_2 .
\end{aligned}
$$

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